Boolean-Valued Multiagent Coalgebraic Logic

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In his article "How True It Is = Who Says It's True' [2], Fitting constructs a modal logic with truth values in the powerset 2^A for some set A of 'agents'. This logic generalises two-valued modal logic and has a natural interpretation suggested by the title of the article: *the truth value of a formula is the set of agents for whom the formula is true.*

This can naturally be made precise through an equation, referred to by Fitting [2] as a Slicing Theorem. For each formula $\varphi \in \mathit{Form}$ of modal logic with a $\Box$-modality, and state $x \in X$ in a labelled Kripke model $\mathbb{M} = (X, R_a)_{a \in A}$, it holds that

$$[\![\varphi]\!]^A_{\mathbb{M}}(x) = \{a \in A \mid x \in [\![\varphi]\!]_{\mathbb{M}_a}\}, \tag{1}$$

where $[\![-]\!]^A_{\mathbb{M}} : \mathit{Form} \to (2^A)^X$ is the semantics of Fitting's logic, and $[\![-]\!]_{\mathbb{M}_a} : \mathit{Form} \to 2^X$ is the semantics of two-valued modal logic over $\mathbb{M}_a = (X, R_a)$. This logic also comes with a natural notion of agent-indexed bisimulations, which is similar in spirit to Equation (1).

We show how coalgebraic logic generalises Fitting-style logics to agent-indexed Set-coalgebras parametric in a functor T. We also generalise to coalgebras over Pos, in which the set A of agents carries an ordering of 'expertise' (cf. [1]) or 'ability'.

Recall the (dual) adjunction between Set and BA, consisting of contravariant functors $P : \mathsf{Set} \to \mathsf{BA}$ (taking powersets) and $S : \mathsf{BA} \to \mathsf{Set}$ (taking ultrafilters). Let T and L be functors as indicated in

$$T \circlearrowright \mathsf{Set} \underset{S}{\overset{P}{\rightleftarrows}} \mathsf{BA} \circlearrowleft L \tag{2}$$

The set of formulas of the coalgebraic modal logic is given by the initial L-algebra I. Given a T-coalgebra (X, ξ), we *algebraify* it into an L-algebra $(PX, \delta_X \circ P\xi)$ by way of a so-called *one-step semantics* $\delta : LP \Rightarrow PT$. By initiality, we then get the semantics $[\![-]\!]_\xi : I \to PX$.

To extend this to Fitting-style agent-indexing, we extend this adjunction to the (co-)Kleisli categories formed from the Set-comonad $A \times (-)$ and the BA-monad $(-)^A$. We denote their corresponding (co-)Kleisli categories by $A\mathsf{Set}$ and $A\mathsf{BA}$.

The adjunction given by P and S lifts to a corresponding adjunction $\bar{P}, \bar{L}$ for $A\mathsf{Set} - A\mathsf{BA}$ by way of obvious natural transformations $P(A \times (-)) \Rightarrow (PX)^A$ and $A \times S(-) \Rightarrow S((-)^A)$. The functors L and T lift to $\bar{L}$ and $\bar{T}$ on $A\mathsf{BA}$ and

$A\mathsf{Set}$ by way of distributive laws $L((-)^A) \Rightarrow (L-)^A$ and $A\times T(-) \Rightarrow T(A\times(-))$. The former is guaranteed to exist here by the universal property of the product $(LB)^A \cong \prod_{a\in A} LB$; we are unsure whether the latter generally exists, although we have not found a T for which this is not the case. Note that all of these lifted functors are defined on objects as the original functors.

Having extended (2) to $A\mathsf{Set}$ and $A\mathsf{BA}$, we can now proceed as before, through the initial $\bar{L}$-algebra I (which has the same underlying set as that of L) and one-step semantics $\bar{\delta} : \bar{L}\bar{P} \Rightarrow \bar{P}\bar{T}$ defined by putting $(\bar{\delta}_X)_a = \delta$ for each $a \in A$, where $(\bar{\delta}_X)_a$ is the 'slice' of $\bar{\delta}_X : LPX \to (PTX)^A$ corresponding to a. The algebraification of $\bar{T}$-coalgebras is defined analogously to before. By initiality and the definition of $A\mathsf{BA}$, we then finally get as the semantics a function $[\![-]\!]^A_\xi : I \to (PX)^A$, satisfying (1) in which $\mathbb{M}$ is replaced by the $\bar{T}$-coalgebra ξ. It is easily verified that this logic instantiated with $T = \mathcal{P}$ and with L-algebras being modal algebras gives us precisely Fitting's original logic, together with its corresponding notions of bisimilarity.

This framework generalises beyond the Set-BA adjunction: tracing our steps for the well-known Pos-DL (or Pre-DL) adjunction (and noting that the (co)monads we considered before are given through (co)powers which generalise to this new setting), we obtain *positive* coalgebraic logics with truth values from 2^A in which an analogous version of (1) holds again. In this setting, the set A of agents has an *ordering* $\preceq$, akin to earlier work by Fitting [1]. By definition of the Pos-DL adjunction, the semantics $[\![\varphi]\!]^A_\xi$ must be upward-closed with respect to A: if $a \preceq b$, then $[\![\varphi]\!]^A_\xi(a) \subseteq [\![\varphi]\!]^A_\xi(b)$. As in [1], this allows us to naturally interpret the ordering on A as one of *relative expertise*: if a dominates b in expertise (i.e. $a \preceq b$), then everything considered true by a must also be considered true by b.

This framework is ripe for further generalisations exploring ways of exploiting structure on agents. As noted by [3], including logical operators acting as permutations on the set of agents can drastically increase these logics' expressive power. This could correspond to equipping the set of agents with a symmetric group action. Taking this idea further, one may be interested in adding operations that create and delete agents (by e.g. using a more general presheaf model of agents). Moreover, topologies on agents could be used to account for potentially infinite sets of agents.

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References

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